Heavy repulsion of clusters

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Descriptive dynamics and combinatorics seminar

Joint work with Tasmin Chu, Greg Terlov and Anush Tserunyan.

Bernoulli Percolation

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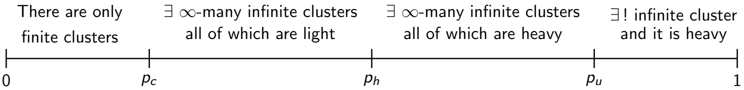

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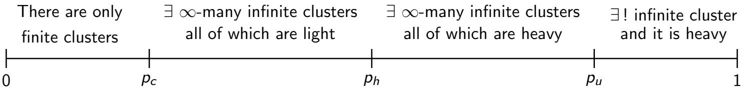

Figure: The four phases of Bernoulli(p) percolation, distinguished by the number of light/heavy infinite clusters.

Question 1 ([HPS99])

For Bernoulli(p) percolation on a quasi-transitive graph, do any two infinite clusters come within distance 1 of each other infinitely often almost surely?

Cluster Repulsion

A negative answer to this question is expected and would follow from the continuity of the two-point function

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Theorem 2 ([Tim06])

Infinite clusters in Bernoulli(p) percolation on unimodular quasi-transitive graphs have finite touching sets almost surely.

Mass Transport

Timár was able to implement the heuristic for repulsion using the **Mass Transport Principle**

Theorem 3

Let G be a unimodular transitive graph and $f : V(G) \times V(G) \to \mathbb{R}_{>0}$ a diagonally invariant function then

$$\sum_{y \in V(G)} f(x,y) = \sum_{y \in V(G)} f(y,x).$$

In particular, this theorem is used by working on a contradictory event and constructing an unbalanced Mass Transport.

Mass Transport (cont.)

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1. Uses **insertion/deletion tolerance** of Bernoulli percolation and **indistinguishability** of its infinite clusters to argue the touching set has at least three ends.
2. Builds a random forest by hand on the set of touching vertices that again has at least 3 ends.
3. Uses a theorem in [BLPS99] which implies that such a forest almost surely has exponential growth

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What is the assumption of unimodularity responsible for?

Nonunimodular graphs

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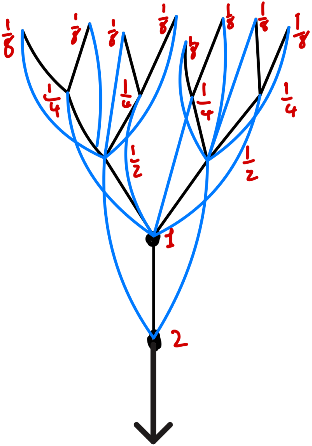
Theorem 5 (Tilted Mass Transport Principle)

Suppose $\Gamma \curvearrowright V$ transitively. Then for any diagonally invariant function $f : V^2 \to \mathbb{R}^+$ and any vertex $o \in V$:

$$\sum_{v \in V} f(o, v) = \sum_{v \in V} f(v, o)\mathbf{w}^o(v).$$

Examples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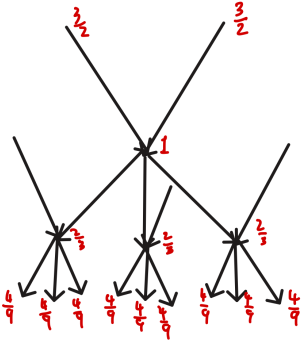
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Result

Theorem 6

Let G be a connected locally finite quasi-transitive nonunimodular graph. Then any two infinite clusters come within distance 1 of each other in a $\mathbf{w}$*-light set of vertices almost surely.*

Here, a set $A \subset V$ is called $\mathbf{w}$-light if $\sum_{x \in A} \mathbf{w}^o(x) < \infty$ and is otherwise called $\mathbf{w}$-heavy.

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Let G be a connected locally finite quasi-transitive nonunimodular graph. Then any two infinite clusters come within distance 1 of each other in a **w***-light set of vertices almost surely.*

Here, a set $A \subset V$ is called **w**-light if $\sum_{x \in A} \mathbf{w}^o(x) < \infty$ and is otherwise called **w**-heavy. This partially answers Question 1, leaving the case of an infinite but **w**-light touching set.

Our strategy

Analogously to [Tim06], we assume that cluster repulsion fails, and construct a non-hyperfinite graph containing vertices in a touching set $\tau(C, C')$. This graph will have exponential weighted growth. We then get a contradiction using a mass transport argument analogous to that of [Tim06].

Constructing such a non-hyperfinite graph comprises the bulk of the argument.

An unbalanced mass transport

Define a mass transport as follows.

- For every $x, y \in V$, if x and y are in the same infinite cluster C and have distance $\leq \alpha$ in G (for some previously fixed α), take UAR a path

$$x = x_1, x_2, \ldots, x_{m-1}, x_m = y$$

  of minimal length in C between them.
- Let x send mass $1/k^2$ to x_k.

The total mass *sent out* by x is $\leq d^\alpha \cdot \sum_n \frac{1}{n^2} < \infty$, where d is the maximal degree of G.

An unbalanced mass transport

Let $o \in V$. If Theorem 2 fails, then with positive probability, there are clusters C, C' and $\alpha > 0$ such that $o \in C$, there is a tree $\phi \subseteq C$ with labeled exponential growth, and every x in ϕ α-touches C'.

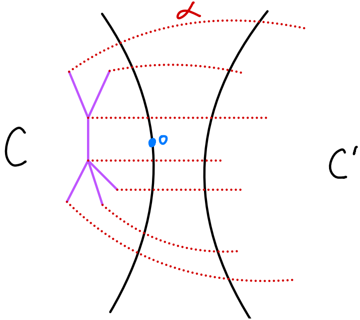
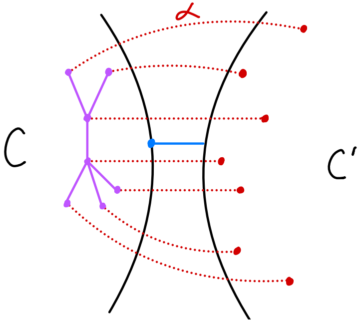
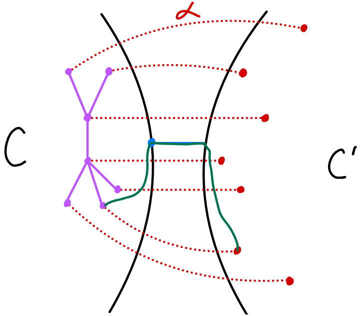


Figure: The α-paths between ϕ and C'

Then w.p.p. the expected mass *received* by o is infinite:
Add an edge between o and its neighbour o' in C'.



This forms a new cluster C''. Then mass needs to be sent from each purple vertex to its corresponding red vertex via a path within C''; this path must go through the blue edge.



Since ϕ has exponential growth, there exists $c > 1$ such that for all n large enough, ϕ has $\geq c^n$ vertices at distance n from o. Each of these vertices sends mass $\frac{1}{n^2}$ to o. Hence, o will receive mass $\geq \gamma \sum \frac{c^n}{n^2} = \infty$ (for some $\gamma > 0$).

$\mathbf{w}$-nonvanishing ends

To construct a non-hyperfinite graph, we use a weighted analogue of regular ends introduced in [CTT22, TTD25].

- An end ξ in G is **w-vanishing** if

$$\lim_{v \to \xi} \mathbf{w}(v) = 0,$$

meaning for all $\varepsilon > 0$, there is some neighbourhood U of ξ such that $w(y) < \varepsilon$ for all $y \in U$.

w-nonvanishing ends

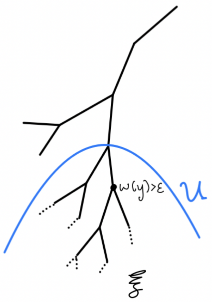


Figure: A nonvanishing end

- We call ξ **w-nonvanishing** if there is $\varepsilon > 0$ such that for all neighbourhoods U of ξ, there is some $y \in U$ with $\mathbf{w}(y) \geq \varepsilon$.

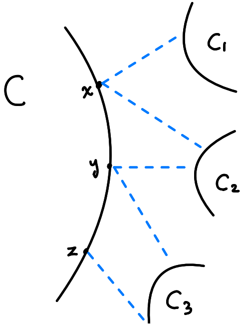
Invariant Random Partitions

Definition 7

A partition τ of V is an **invariant random partition (IRP)** if it is an $\text{Aut}(G)$-invariant probability measure on the set of all partitions of V.

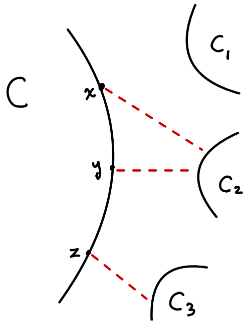
We define an IRP $\hat{\tau}$ as follows.

- Given a Bernoulli(p) configuration ω and $x \in V$, let x pick a cluster at distance 1 from x uniformly at random, if such a cluster exists.
- We define a pair of vertices $\{x, y\}$ to be in the same class of the partition $\hat{\tau}$ if and only if x and y are in the same cluster in ω, and x and y picked the same neighboring cluster.



In this configuration ω, it holds that

$$x \in \tau(C, C_1), \quad x, y \in \tau(C, C_2) \quad \text{and} \quad y, z \in \tau(C, C_3).$$



In the corresponding invariant random partition $\hat{\tau}_\omega$, it holds that

$$x, y \in \hat{\tau}(C, C_2) \quad \text{and} \quad z \in \hat{\tau}(C, C_3).$$

Touching classes

- Note that every $\hat{\tau}$-class is uniquely characterized by a pair of clusters C, C'; we write $\hat{\tau}(C, C')$ to make this explicit.
- While the touching set $\tau(C, C')$ is *not* invariant, the **touching class** $\hat{\tau}(C, C') \subseteq \tau(C, C')$ is invariant. Moreover, $(\hat{\tau}(C, C'), \omega)$ is invariant.

Our strategy

Suppose towards a contradiction that with positive probability, there are clusters C, C' such that $\tau(C, C')$ is heavy.

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3. $\hat{\tau}$ is non-hyperfinite

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$\Rightarrow$ Contradiction! By TMTP.

At least one **w**-nonvanishing end

Lemma 8

Consider a jointly invariant random process $(\omega, \hat{\tau})$, where ω is a configuration of a Bernoulli(p) bond percolation and $\hat{\tau}$ is a partition of ω into the touching classes. Then a.s. for any two distinct clusters C, C' in ω if $\tau(C, C')$ is heavy then C has at least one **w***-nonvanishing end along $\hat{\tau}(C, C')$.*

This follows from a brief subsampling argument: let d be the maximum degree of any vertex in G; then on average, the touching classes will contain at least a $1/d$-proportion of vertices from the touching set, so they will almost surely be heavy.

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≥ 3 **w**-nonvanishing ends

We will use that if Theorem 6 fails, then a.s. every heavy cluster has a heavy touching class with infinitely many other heavy clusters.
This is a corollary of a more general result about touching classes.

Definition 9

Let $\mathcal{N}$ be an invariant property of a subset of vertices of G. We say that a cluster $C \subset \omega$ is an $\mathcal{N}$-**neighbor** of a cluster $C' \subset \omega$, and write $C \sim_{\mathcal{N}} C'$, if the set $\tau(C, C')$ of vertices in C that are within unit distance of C' satisfies the property $\mathcal{N}$. We say that the property is **symmetric** if $C \sim_{\mathcal{N}} C'$ always implies that $C' \sim_{\mathcal{N}} C$.

E.g. being nonempty, finite, heavy, or infinite-light.

Neighbouring properties

Lemma 10 (Neighbouring properties)

For any symmetric invariant property $\mathcal{N}$ on G, for $\mathbb{P}_p$-a.e. configuration ω, either every heavy cluster in ω has no $\mathcal{N}$-neighboring clusters or every heavy cluster in ω has infinitely many $\mathcal{N}$-neighboring clusters.

Corollary 11

Every heavy cluster has a heavy touching set with infinitely many other heavy clusters, or no other heavy clusters, $\mathbb{P}_p$-a.s.

Neighbouring properties

Proof sketch. Let $\mathcal{N}$ be an invariant symmetric property and let $H_{\mathcal{N}}$ be the corresponding graph, that is, a graph whose vertices are the clusters of G, with edges between every pair of $\mathcal{N}$-neighbors. By the indistinguishability of heavy clusters of Bernoulli(p) percolation [Tan19, Theorem 1.1] $H_{\mathcal{N}}$ is d-regular a.s., for some possibly infinite d.
We proceed by case analysis on the value of d, and conclude that $d = 0$ or $d = \infty$ almost surely.

$\boxtimes$

Lemma 10 also allows us to show the following, answering a question posed in [HPS99].

Proposition 12 (Relentless merging of heavy clusters)

Let G be an infinite, locally finite, connected, quasi-transitive nonunimodular graph. Then $\mathbb{P}$-a.s. for every $p_1 \in (p_c, p_u)$ and $p_2 \in (p_h, 1]$ such that $p_1 < p_2$, every infinite p_2-cluster contains infinitely many infinite p_1-clusters.

≥ 3 **w**-nonvanishing ends

Proposition 13

Consider a jointly invariant random process $(\omega, \hat{\tau})$, where ω is a configuration of a Bernoulli(p) bond percolation, and $\hat{\tau}$ is a partition of ω into the touching classes. Suppose the conclusion of Theorem 6 does not hold, then with positive probability there is a cluster that has ≥ 3 $\mathbf{w}$-nonvanishing ends along some $\hat{\tau}$-class, none of which are isolated.

Proof. We proceed by case analysis.

≥ 3 $\mathbf{w}$-nonvanishing ends

First, suppose that with positive probability, there exist clusters C_1, C_2, D_1, D_2 such that

$$\mathbf{w}(\hat{\tau}(C_1, D_1)) = \mathbf{w}(\hat{\tau}(C_2, D_2)) = \infty$$

and the remaining touching *sets* (which contain corresponding touching classes) between these clusters are empty.

This is exactly the setting of [Tim06, Lemma 2.3]; one can show that wpp there exist $C, C' > 0$ such that $\hat{\tau}(C, C')$ has ≥ 3 $\mathbf{w}$-nonvanishing ends.

≥ 3 **w**-nonvanishing ends

Suppose that a.s. no such clusters C_1, C_2, D_1, D_2 exist. Let $\mathcal{N}$ represent the heavy neighbouring relation between clusters; then every heavy cluster has infinitely many $\mathcal{N}$-neighbours. We insert edges to create two clusters C, C' such that C has ≥ 3 **w**-nonvanishing ends along $\hat{\tau}(C, C')$.

$\boxtimes$

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3. $\hat{\tau}$ is non-hyperfinite
4. $\hat{\tau}(C, C')$ exhibits exponential weighted growth

$\Rightarrow$ Contradiction! By TMTP.

Cluster graphing

In order to pass from non-hyperfiniteness to weighted exponential growth, we pass through the **cluster graphing**.

Definition 14 ([JKL02],[Gab05],[CTT22])

Let $\mathbf{P}$ be the law of a Γ-invariant random graph on V. A Borel graph $\mathcal{G}$ on a standard probability space (X_0, μ_0) is a **cluster graphing** of $\mathbf{P}$ if $X_0 = (X \times V)/\Gamma$ and $\mu_0 = (\pi_0)_*\mu$ for some pmp action $\Gamma \curvearrowright (X, \mu)$ and $(\omega_{\mathcal{G}})_*\mu = \mathbf{P}$.

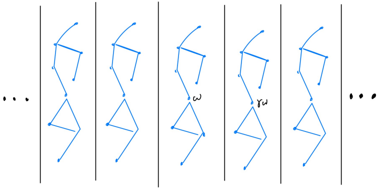


Figure: Caption

Cluster graphing

Orbit equivalence		Percolation
X	$\overset{\psi}{\leftrightarrow}$	$\Gamma\backslash(X \times V)$
$\gamma^{-1} \cdot x$		$[(x, \gamma \cdot \rho)]$
$x \in X_\infty$		$\rho^{\pi(x)}\infty$
Borel subset		vertex property
R_{cl}-class		cluster
R_{cl}-invariant		cluster property
ergodicity of R	$\simeq$	indistinguishability
ϕ s.t. $\mathrm{gr}(\phi) \subset R_{cl}$		rerooting
$\phi \in [R]$		vertex-bijective rerooting
asymptotically R_{cl}-invariant		asymptotic cluster property
strong ergodicity of R	$\simeq$	strong indistinguishability
graphing		graph structure

Credit: Sébastien Martineau

Nonhyperfinite graphings

Take a pair $(\mathcal{G}^{cl}, \mathcal{S})$ with $\mathcal{G}^{cl}$ is a cluster graphing that factors onto $(\omega, \hat{\tau}_\omega)$ and $\mathcal{S}$ is the subequivalence relation induced by $\hat{\tau}_\omega$.

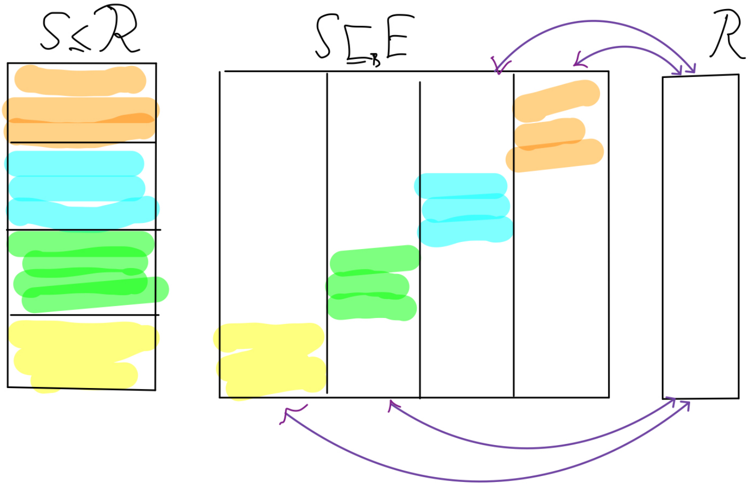
Proposition 15 ([CTT22, Proposition 3.24])

Let $\mathcal{G}$ be a locally finite mcp graph on a standard probability space (X, μ) and let $\mathbf{w}_\mu$ be the Radon–Nikodym cocycle of its connectedness relation with respect to μ. If a.e. $\mathcal{G}$-component has ≥ 3 $\mathbf{w}$-nonvanishing ends then $\mathcal{G}$ is μ-nowhere hyperfinite.

Proposition 16 ([CK18, Proposition 5.3])

Suppose $\mathcal{S} \leq \mathcal{R}$ is a pair of cBers. There is a cBer E such that $\mathcal{S}$ embeds into E as a complete section and there is a class-bijective homomorphism $E \twoheadrightarrow \mathcal{R}$.

Relativization



Nonhyperfinite graphings

Combining these two results and pushing the graph structure of $\mathcal{G}^{cl}$ onto the new equivalence relation, we get a genuine nonhyperfinite mcp graph.

Theorem 17 ([Kai97])

Let $\mathcal{G}$ be an mcp graph on (X, μ) of uniformly bounded degree. Then $\mathcal{G}$ is hyperfinite if and only if for any measurable $B \subseteq X$ of μ-positive measure, $\Phi_x^{\mathbf{w}_\mu}(\mathcal{G}|_B) = 0$ for μ-a.e. $x \in X$.

$$\Phi_x^{\mathbf{w}_\mu}(\mathcal{G}) = \inf \left\{ \frac{\mathbf{w}_\mu^x(\partial_{\mathcal{G}} F)}{\mathbf{w}_\mu^x(F)} : F \subseteq [x]_{\mathcal{G}} \text{ finite and nonempty} \right\}$$

In fact, this machinery allows us to conclude that $\hat{\tau}_\omega$ is nonhyperfinite as an invariant measure on $2^{V \times V}$ and the positive weighted Cheeger constant yields the weighted exponential growth we need.

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